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*The General Transformation Theory of Differential Elements.**

BY EDWARD KASNER.

The simplest type of differential element is the lineal element of the plane, defined by (x, y, y') or (x, y, p) . The general element transformation is obtained by setting x_1, y_1, p_1 equal respectively to three arbitrary functions of x, y, p . Contact transformations are those special element transformations which convert every union into a union, a union (*Verein*) consisting of the ∞^1 elements of a curve or point. The general transformation will usually disrupt a union; that is, the new ∞^1 elements will not be united. Our first result is that for an arbitrary transformation (not of the contact class) there exist precisely ∞^2 unions which are preserved, that is, converted into unions. It follows that if a transformation preserves more than a doubly infinite number of unions, it must be a contact transformation.

We consider next transformations of the direction elements of space (x, y, z, y', z') . Here, intermediate between the general case preserving ∞^4 unions, and the contact case preserving all unions, we find a type preserving ∞^∞ unions. These are solutions of a certain form of Monge equation of the second order which is characterized geometrically by a property equivalent to that stated in Meusnier's theorem on the curves of a surface. The intermediate type may be obtained in explicit form and involves three arbitrary functions.

The discussion of transformations of surface elements (x, y, z, p, q) , taken up in Section III, is more complicated. In the most general case no two-dimensional unions (surfaces, curves or points) are preserved. In the contact type, of course, all unions are preserved. In between we find types preserving a single, double or triple infinity, and also a type where the system of unions involves an arbitrary function, being defined by a pair of equations in involution. The last type,

* The main results of this paper were presented before the American Mathematical Society, April 30, 1904.

which is the most interesting as being the nearest to the contact type, depends upon three arbitrary functions of five arguments. Among the special results, it is shown that if every point and every plane is converted into a union, the transformation is necessarily one of contact. It follows, for example, that no element transformations beyond those contained in Lie's fifteen-parameter group can convert every sphere into a sphere.

With respect to one-dimensional unions (strips), all transformations not of the contact type preserve a family involving two arbitrary functions.

In Section IV we consider briefly the differential elements of n -th order in the plane. Usually, no unions are preserved, but the number may vary in special cases from ∞^1 to ∞^{n+1} . If more than ∞^{n+1} , then all are preserved.

I. *Direction Elements in the Plane.*

The general transformation is of the form

$$x_1 = X(x, y, p), \quad y_1 = Y(x, y, p), \quad p_1 = P(x, y, p), \quad (1)$$

where the three functions are assumed to be uniform, to possess continuous partial derivatives, and to have a non-vanishing Jacobian.

Consider now two united elements (x, y, p) and $(x + dx, y + dy, p + dp)$, so that $dy - p dx = 0$. The condition that the transformed elements (x_1, y_1, p_1) and $(x_1 + dx_1, y_1 + dy_1, p_1 + dp_1)$, where

$$dx_1 = X_x dx + X_y dy + X_p dp, \text{ etc.},$$

shall also be united is $dy_1 - p_1 dx_1 = 0$, or

$$(Y_x - PX_x) dx + (Y_y - PX_y) dy + (Y_p - PX_p) dp = 0,$$

which becomes, on replacing dy by $p dx$,

$$\{Y_x + p Y_y - P(X_x + p X_y)\} dx + \{Y_p - P X_p\} dp = 0. \quad (2)$$

If (1) is a contact transformation, all united elements remain united, so that (2) is satisfied identically; we then have the familiar necessary and sufficient conditions

$$Y_x + p Y_y - P(X_x + p X_y) = 0, \quad Y_p - P X_p = 0. \quad (3)$$

Omitting this case, we may solve (2) for dp/dx , that is d^2y/dx^2 or y'' , and write

$$y'' = \frac{P(X_x + p X_y) - (Y_x + p Y_y)}{Y_p - P X_p}. \quad (4)$$

Hence if (1) is not a contact transformation, there is for each element a unique united element such that the two transformed elements are also united.

Our condition (4) is a differential equation of the second order

$$y'' = \omega(x, y, y'), \quad (4')$$

and therefore defines a family of ∞^2 curves. In the particular case where $Y_p - PX_p$ vanishes, (4') takes the form $y'' = \infty$ and defines the ∞^2 points of the plane.

For a general element transformation (1) (not a contact transformation), there exists always a doubly infinite system of unions which are alone converted into unions. The differential equation of the system is (4).

As an important corollary, we have:

If a transformation converts more than ∞^2 unions into unions, the same must be true for all unions, and hence it is a contact transformation.

Consider, as an application, the determination of the group of element transformations which convert every circle into a circle. Since the number of circles is ∞^3 , it follows that the result is merely Lie's ten-parameter group of contact transformations.

On the other hand, the group of element transformations which convert straight lines into straight lines is larger than the corresponding group of contact transformations. The latter involves two arbitrary functions of two arguments, while the former involves an additional arbitrary function of three arguments. Since all differential equations of the second order are equivalent under contact transformation, a similar remark applies to all doubly infinite systems of curves. In case of the ∞^2 point unions, the element transformations are obviously

$$x_1 = \phi(x, y), \quad y_1 = \psi(x, y), \quad p_1 = P(x, y, p),$$

while the contact transformations are

$$x_1 = \phi(x, y), \quad y_1 = \psi(x, y), \quad p_1 = \frac{\psi_x + p\psi_y}{\phi_x + p\phi_y}.$$

We note finally that if a transformation converts every point and every straight line into a union, it must be of the contact category.

II. Direction Elements in Space.

A direction element in space is defined by five coordinates (x, y, z, y', z') ; we employ also the notation (x, y, z, p, q) when more convenient.* The general

* In the next section p, q will be used in the customary manner to denote partial derivatives of z with respect to x and y .

transformation is

$$x_1 = X(x, y, z, p, q), \quad \dots, \quad q_1 = Q(x, y, z, p, q). \quad (5)$$

Lie showed that the only transformations of this sort which convert *all* curves (or unions) into curves (or unions) are the extended point transformations. We inquire whether the general transformation (5) converts *some* unions into unions.

Using the two conditions for a union,

$$dy - p dx = 0, \quad dz - q dx = 0,$$

and proceeding as in I, we find

$$\begin{aligned} (PX_p - Y_p)dp + (PX_q - Y_q)dq &= \{Y_x + pY_y + qY_z - P(X_x + pX_y + qX_z)\}dx, \\ (QX_p - Y_p)dp + (QX_q - Y_q)dq &= \{Z_x + pZ_y + qZ_z - Q(X_x + pX_y + qX_z)\}dx. \end{aligned} \quad (6)$$

Omitting the case of extended point transformations, we see that these will not both hold identically. Assuming first that the determinant

$$\begin{vmatrix} PX_p - Y_p & PX_q - Y_q \\ QX_p - Y_p & QX_q - Y_q \end{vmatrix} \quad (7)$$

does not vanish, we may solve (6) in the form

$$y'' = g(x, y, z, y', z'), \quad z'' = h(x, y, z, y', z'), \quad (6')$$

and have then a pair of equations defining ∞^4 curves.

On the other hand, if the above determinant vanishes identically, but the equations (6) are not equivalent, it follows that dx must vanish and dp and dq are connected by a single relation

$$(PX_p - Y_p)dp + (PX_q - Y_q)dq = 0. \quad (8)$$

Thus, for any fixed values of x, y, z , we find q as a function of p and a constant of integration. Our unions are then of special type, consisting of elements with a common point forming a cone. The total number of these is as before ∞^4 .

We must consider finally the case where the equations (6) are equivalent to each other; that is, where

$$\left\| \begin{array}{ccc} PX_p - Y_p & PX_q - Y_q & Y_x + pY_y + qY_z - P(X_x + pX_y + qX_z) \\ QX_p - Y_p & QX_q - Y_q & Z_x + pZ_y + qZ_z - Q(X_x + pX_y + qX_z) \end{array} \right\| = 0. \quad (9)$$

The single equation to be satisfied is then of the form

$$E y'' + F z'' + G = 0, \quad (10)$$

where E, F, G involve only x, y, z, y', z' . Such a Monge equation of the second order is satisfied by a family of curves involving an arbitrary function; in fact, upon any surface there will be ∞^2 integral curves. It is easily shown from the linear form of (10) that the integral curves have the *Meusnier property*; namely, the osculating circles of those curves of the family which pass through a given point in a given direction generate a sphere. Conversely, (10) represents the most general family for which this is true.*

The case where conditions (9) hold thus leads to a type of element transformation intermediate between the completely arbitrary type and the contact type. Our result may be stated as follows:

All transformations (5) of the direction elements of space may be classified with respect to the number of unions preserved into these types:

Type 1°. Extended point transformations. All unions preserved.

Type 2°. Transformations fulfilling conditions (9), but not extended point transformations. The family of unions preserved involves an arbitrary function: It is defined by a Monge equation of the second order of the form (10), characterized by the possession of the Meusnier property.

Type 3°. Transformations not fulfilling conditions (9). The number of unions preserved is ∞^4 .

The last is, of course, the general case, involving five arbitrary functions of three arguments. Type 1° involves three arbitrary functions of three arguments. The intermediate type 2° involves three arbitrary functions of five arguments: we may in fact take the functions X, Y, Z at random and still satisfy conditions (9), the functions P and Q being then in general† uniquely determined.

III. Surface Elements of Space.

A surface element (x, y, z, p, q) consists of a point (x, y, z) and an incident plane, the direction cosines of the normal being proportional to $p : q : -1$. The condition for united elements is

$$dz = p dx + q dy.$$

* Cf. the author's note "The Inverse of Meusnier's Theorem," *Bull. Amer. Math. Soc.*, Vol. XIV (1908), pp. 461-465.

† Unless certain determinants vanish.

In connection with a surface $z = f(x, y)$, we employ the usual notation $p = f_x$, $q = f_y$, $r = f_{xx}$, $s = f_{xy}$, $t = f_{yy}$, so that

$$dp = r dx + s dy, \quad dq = s dx + t dy.$$

Applying now the general element transformation

$$x_1 = X(x, y, z, p, q), \quad \dots, \quad q_1 = Q(x, y, z, p, q), \quad (11)$$

we have, as the condition that the transformed elements form a union,

$$dz_1 = p_1 dx_1 + q_1 dy,$$

or

$$\alpha dx + \beta dy + \gamma dz + \delta dp + \varepsilon dq = 0, \quad (12)$$

where

$$\left. \begin{aligned} \alpha &= Z_x - PX_x - QY_x, & \beta &= Z_y - PX_y - QY_y, & \gamma &= Z_z - PX_z - QY_z, \\ \delta &= Z_p - PX_p - QY_p, & \varepsilon &= Z_q - PX_q - QY_q. \end{aligned} \right\} \quad (12')$$

Substituting the values of dz , dp , and dq given above, and equating the coefficients of dx and dy in the result to zero, we find the fundamental pair of equations

$$Ar + Bs + C = 0, \quad As + Bt + D = 0, \quad (13)$$

where the coefficients are

$$A = \delta, \quad B = \varepsilon, \quad C = \alpha + pr, \quad D = \beta + qr, \quad (13')$$

and thus involve only x, y, z, p, q .

If our element transformation (11) converts any surfaces into unions, such surfaces must satisfy the pair of equations (13).

In the case of contact transformations the equations become identities: the familiar conditions for a contact transformation are in fact

$$A = B = C = D = 0. \quad (14)$$

We exclude this case from the remaining discussion.

A pair of partial differential equations of the second order involving one function of two independent variables will in general be inconsistent. In special cases there may be a finite number of solutions; a solution involving one, two, three or four arbitrary constants; or finally, when the equations are in involution, the solution may involve an infinity of constants, that is, an arbitrary function.

Applying the general theory to our pair (13), we differentiate each of the equations with respect to x and to y , and form with the coefficients of the four resulting equations, which are linear in the partial derivatives of third order, the following matrix:

$$\begin{vmatrix} A & B & 0 & 0 & r \frac{dA}{dx} + s \frac{dB}{dx} + \frac{dC}{dx} \\ 0 & A & B & 0 & r \frac{dA}{dy} + s \frac{dB}{dy} + \frac{dC}{dy} \\ 0 & A & B & 0 & s \frac{dA}{dx} + t \frac{dB}{dx} + \frac{dD}{dx} \\ 0 & 0 & A & B & s \frac{dA}{dy} + t \frac{dB}{dy} + \frac{dD}{dy} \end{vmatrix}, \quad (15)$$

where, for example,

$$\frac{dA}{dx} \equiv A_x + pA_z + rA_p + sA_q, \quad \frac{dA}{dy} \equiv A_y + qA_z + sA_p + tA_q. \quad (15')$$

The condition for involution is that all the determinants of fourth order in the above matrix shall vanish in virtue of (13). Only two independent relations are found, namely

$$\left. \begin{aligned} CD(A_q - B_p) - BC(A_y + qA_z - D_p) + AD(B_x + pB_z - C_q) + AB(C_y - D_x + qC_z - pD_z) = 0, \\ (AC + BD)(A_q - B_p) - B^2(A_y + qA_z - D_p) + A^2(B_x + pB_z - C_q) \\ + AB(B_y - A_x + qB_z - pA_z + C_p - D_q) = 0. \end{aligned} \right\} \quad (16)$$

Our result is then as follows:

If an element transformation (11), without being of the contact type, converts an infinitude of surfaces involving an arbitrary function into unions, then the pair of equations (13) will be in involution and hence relations (16) will hold.

The type characterized by relations (16) is thus the nearest to the contact type, in the sense that it preserves the maximum number of unions, without preserving all. The manifold of surfaces is such that ∞^1 pass through an arbitrarily selected curve.

If any pair of equations of the second order possess more than ∞^4 solutions, they will, according to standard theory, have ∞^∞ solutions, and hence be in involution. In the case of our equations (13) we obtain a stronger result. We notice that the determinant formed from the first four columns of the matrix (15) (that is, from the coefficients of the third derivatives in the set of four linear equations referred to above) vanishes identically.* The assumption of ∞^4 common solutions requires that the four linear equations shall be consistent. In the present case this is impossible unless *all* the determinants in (15) vanish; that is, the case of involution. Therefore:

* This means that the equations (13) always have one set of characteristics in common.

If an element transformation converts ∞^4 surfaces into unions, the equations (13) are necessarily in involution; that is, ∞^∞ surfaces are so converted. The transformation thus belongs to the type described in the preceding theorem (or else it is of the contact type).

We next consider transformations preserving a triply infinite system of unions. This type, which is of course more general than the preceding type, can be determined explicitly from general principles. We observe first that any ∞^3 unions (surfaces, curves or points) may be converted into any other ∞^3 unions by a contact transformation. In particular, any ∞^3 unions may be converted into points. The most general element transformation which converts every point union into itself, is obviously of the form

$$x_1 = x, \quad y_1 = y, \quad z_1 = z, \quad p_1 = P(x, y, z, p, q), \quad q_1 = Q(x, y, z, p, q), \quad (17)$$

involving two arbitrary functions P and Q . Denote this infinite group by G , and let Γ_1, Γ_2 denote any two contact transformations. Then

$$\Gamma_1 G \Gamma_2 \quad (18)$$

converts the ∞^3 unions which become points under Γ_1 into the ∞^3 unions which are obtained by applying Γ_2 to points. All transformations of the required type are included under the form (18). Remembering that a general contact transformation involves an arbitrary function of five arguments, we state our result as follows:

All element transformations which preserve as many as ∞^3 unions are included in the form $\Gamma_1 T \Gamma_2$, where Γ_1, Γ_2 are arbitrary contact transformations, and T is any member of the group (17) which converts every point union into itself. The result involves four arbitrary functions of five arguments.

Consider in particular the cases where the ∞^3 unions are the points or the planes of space. If every point is to be converted into a union, equation (12) must be satisfied on substituting $dx = dy = dz = 0$. It follows that δ and ϵ must vanish; that is, in the notation of (13'), $A = 0, D = 0$. On the other hand, if every plane is to become a union, our fundamental equation (13) must be satisfied when $r = s = t = 0$. The requisite conditions are $C = 0, D = 0$. Remembering that the vanishing of the quantities A, B, C, D is the criterion for a contact transformation, we derive this result:

If a transformation of the surface elements of space converts both points and planes into unions, it must be a contact transformation.

This conclusion does not hold for two sets of ∞^3 unions selected at random. For it may happen that the two sets are part of a system of ∞^∞ unions defined by a pair of equations (13) in involution, in which case there will exist transformations not of the contact type. Similar remarks apply, of course, to systems of unions involving four or more parameters.

For certain systems, however, only contact transformations exist. Consider the ∞^4 spheres. Their differential equations are

$$pqr - (1 + p^2)s = 0, \quad (1 + q^2)s - pqt = 0. \quad (19)$$

If these are equivalent to a set of type (13), we must have $A = B = C = D = 0$; that is, a contact transformation. This result also follows directly from that stated above, since spheres include points and planes as limiting cases. Recalling the equivalence of spheres and straight lines, we have:

No element transformations beyond the contact type can convert every sphere, or every straight line, into a union.

In particular, the known fifteen-parameter group of sphere transformations includes all element transformations which turn spheres into spheres.

We return to the consideration of equation (12), defining all unions which are preserved under (11). In the general case where the union (of ∞^2 elements) is a surface, we obtain the partial differential equations (13). The case of a curve regarded as a union of ∞^2 elements may be treated as a limiting case of a surface. The direct treatment however is as follows:

For the surface elements of a curve whose point equations are $y = y(x)$, $z = z(x)$, we have

$$p = z' - qy'. \quad (20)$$

Substituting in (12), we obtain an identity in dx, dq giving two conditions

$$\left. \begin{aligned} \alpha + \beta y' + \gamma z' + \delta(z'' - qy'') &= 0, \\ \varepsilon - y'\delta &= 0. \end{aligned} \right\} \quad (21)$$

These equations are, on eliminating p by means of (20), to become identities in q . Usually no solutions will exist; for certain element transformations there may be a triply infinite system of curves or even an infinitude involving an arbitrary function, in which case the family is defined by a Monge equation $f(x, y, z, y', z') = 0$. A detailed discussion is rendered unnecessary by observing that, to the curves turned into unions by any given transformation T correspond surfaces preserved by the related transformation RT , where R denotes polarity with respect to a quadric surface.

Summarizing our results, we have the following

THEOREM. *With respect to the number of unions (of ∞^2 elements) which are converted into unions, transformations of the surface elements of space may be classified as follows:*

Type 1°. Contact transformations. All unions preserved.

Type 2°. Transformations, other than of type 1°, preserving ∞^∞ unions. The unions are then defined by an involution system.

Type 3°. Transformations, not of the previous types, preserving ∞^3 unions.

Type 4°. Transformations preserving fewer than ∞^3 unions. The number may be ∞^2 , ∞^1 , a discrete number, or none at all, the last being the usual case.

Type 1° is characterized by four conditions $A = B = C = D = 0$, and thus involves one arbitrary function of five arguments. Type 2° is characterized by the two conditions (16), and depends on three arbitrary functions. Type 3°, represented explicitly by (18), involves four arbitrary functions. Type 4°, being the general case, involves five arbitrary functions.*

We consider finally the effect of any transformation on one-dimensional unions. The general case of such a union consists of the ∞^1 surface elements touching a given surface along a given curve, thus forming a *strip*. A special case arises when the ∞^1 elements have a common point, thus enveloping a cone, which may in particular shrink to a line. The strips preserved must satisfy the equations

$$\alpha + \beta y' + \gamma z' + \delta p' + \varepsilon q' = 0, \quad p + q y' - z' = 0, \quad (22)$$

where y, z, p, q are considered as unknown functions of x . We easily derive the following results:

The only transformations preserving all one-dimensional unions are the contact transformations.

For an arbitrary element transformation, the number of strips preserved is ∞^∞ , involving two arbitrary functions.

IV. *Elements of Higher Order.*

Confining our attention to the plane, we consider the general transformation of differential elements of the n -th order

$$\left. \begin{aligned} \bar{x} &= X(x, y, y', \dots, y^{(n)}), & \bar{y} &= Y(x, y, y', \dots, y^{(n)}), \\ \bar{y}' &= Y_1(x, y, y', \dots, y^{(n)}), & \dots, & \bar{y}^{(n)} = Y_n(x, y, y', \dots, y^{(n)}). \end{aligned} \right\} \quad (23)$$

* This remains true if we take only those transformations which do not preserve any unions.

The conditions for a union are

$$dy = y' dx, \quad dy' = y'' dx, \quad \dots, \quad dy^{(n-1)} = y^{(n)} dx.$$

If a curve is to be converted into a union, we find that it must satisfy the following n differential equations of the $(n+1)$ -th order:

$$y^{(n+1)} = \omega_k(x, y, y', \dots, y^{(n)}) \quad (k = 1, 2, \dots, n), \quad (24)$$

where

$$\omega_k \equiv \frac{\frac{dY_{k-1}}{dx} - Y_k \frac{dX}{dx}}{\frac{\partial Y_{k-1}}{\partial y^{(n)}} - Y_k \frac{\partial X}{\partial y^{(n)}}}. \quad (24')$$

The equations (24) become identities only in the case of extended contact transformations. *In general the equations will be inconsistent, so that no unions will be preserved. In intermediate types there may be ∞^j curves, where j varies from 1 to $n+1$ inclusive.*

If a transformation (23) preserves more than ∞^{n+1} unions, it must preserve all and hence be of the contact type. The type nearest to the contact type, preserving ∞^{n+1} curves, may easily be determined explicitly in terms of three arbitrary functions. We note that with respect to the group (23) all differential equations of the $(n+1)$ -th order are equivalent.

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